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A Class of p-valent functions defined by differential operator

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Abstract:

In the present paper, we introduce the class $\mathrm{A}(\alpha,\beta,\lambda,\delta,n,p,A,B)$ of p-valent analytic functions in the open unit disk U. We investigate some inclusion properties, coefficient bounds, distortion theorem, ε-neighborhoods and partial sums. Also we obtain integral representation, weighted and arithmetic mean.

Keywords: Differential subordination, p-valent functions, differential operator.

1. Introduction:

Let $H(u)$ the class of analytic functions in the open unit disk $U=\{z\in\mathbb{C}:|z|<1\}$ and let $H(a,p)$ the class of function $f\in$ h of the form

$$f(z)=a+a_pz^p+a_{p+1}z^{p+1}+\dots,\ \left(a\in\mathbb{C};\ p\in\mathbb{N}=\{1,2,3,\dots\}\right) \quad . \quad (1.1)$$

Also, let $\mathbf{A}(p)$ be the subclass of h consisting of functions of the form:

$$f(z)=z^p+\sum_{k=m}^{\infty}a_kz^k,\ \left(m\ge p,\ p\in\mathbb{N}=\{1,2,\dots\}\right). \quad (1.2)$$

Definition 1.1: Let $f, g \in$ h, if there exist a Schwarz function w analytic in U with $w(0)=0$ and $|w(z)|<1, z\in U$ such that $f(z)=g(w(z))$, then the function f is called subordinate to g, or g is called superordinate to f, in such a case we write $f \prec g$ or $f(z)\prec g(z),\ z\in U$. If g is univalent in U, then $f\prec g$ if and only if $f(0)=g(0)$ and $f(U)\subset g(U)$.

In [9], we introduced and studied the differential operator $D^n(\alpha,\beta,\lambda,\delta)f(z)$,

for a functions $f(z)$ in the class A as the following:

$$D^0 f(z)=f(z),$$

$$D^1(\alpha,\beta,\lambda,\delta)f(z)=\left[1-(\lambda-\delta)(\beta-\alpha)\right]f(z)+(\lambda-\delta)(\beta-\alpha)zf'(z)$$

$$=z+\sum_{k=2}^{\infty}\left[(\lambda-\delta)(\beta-\alpha)(n-1)+1\right]a_k z^k$$

$$\vdots$$

$$D^n(\alpha,\beta,\lambda,\delta)f(z)=z+\sum_{k=2}^{\infty}\left[(\lambda-\delta)(\beta-\alpha)(n-1)+1\right]^n a_k z^k, \tag{1.4}$$

for $\alpha\ge 0,\beta\ge 0,\lambda\ge 0,\delta\ge 0,\lambda>\delta,\beta>\alpha,\ n\in\{0,1,2,...\}$.

Now, we define the operator $D^n(\alpha,\beta,\lambda,\delta)f(z)$ in (1.4) of a function $f(z)\in \mathbf{A}(p)$ given by (1.1) as

$$D_p^n(\alpha,\beta,\lambda,\delta)f(z)=z^p+\sum_{k=m}^{\infty}\left[\frac{(\lambda-\delta)(\beta-\alpha)k+p}{p}\right]^n a_k z^k \tag{1.5}$$

for $\alpha\ge 0,\beta\ge 0,\lambda\ge 0,\delta\ge 0,\lambda>\delta,\beta>\alpha,\ n\in\{0,1,2,...\},\ m\ge p,\ p\in\mathbb{N}$.

We note that:

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When $\alpha=\delta=0,\ \beta=\lambda=p=1$ it reduces to Sãlãgan differential operator [10]. It reduces to Darus & Ibrahim, when $\alpha=0,\ p=1$[4]. It reduces to Al-oboudi differential operator [1]. when, $\alpha=\delta=0,\ \beta=p=1$.

Definition1.2: Let A and $B\left(-1\le B<A\le 1\right)$ be fixed parameters, we say that a function $f(z)\in$ **A**(p) in the class $f(z)\in \mathrm{A}(\alpha,\beta,\lambda,\delta,n,p,A,B)$, if it satisfies the following subordination condition

$$\left\{\frac{pz^{p}+z\left(D_p^n(\alpha,\beta,\lambda,\delta)f(z)\right)'}{2z^{p}}\right\}\prec\frac{1+Az}{1+Bz},\qquad \left(z\in U\setminus\{0\}\right). \tag{1.3}$$

By the definition of differential subordination (1.3) , is equivalent to the following condition

$$\left|\frac{pz^{p}+z\left(D_p^n(\alpha,\beta,\lambda,\delta)f(z)\right)'}{(B-2A)pz^{p}+Bz\left(D_p^n(\alpha,\beta,\lambda,\delta)f(z)\right)'}\right|<1$$

2. Coefficient bounds

Theorem 2.1: Let the function $f(z)$ of the form (1.1), be in $f(z)\in \mathrm{A}(p)$. Then the function $f(z)$ belongs to the class $f(z)\in \mathrm{A}(\alpha,\beta,\lambda,\delta,n,p,A,B)$ if and only if

$$\sum_{k=m}^{\infty} k(1-B)\left((\lambda-\delta)(\beta-\alpha)k+p\right)^{n}a_k\le 2(B-A-1)p^{2}, \tag{2.1}$$

where $-1\le B<A\le 1,\ \alpha\ge 0,\beta\ge 0,\lambda\ge 0,\delta\ge 0,\lambda>\delta,\beta>\alpha,\ n\in\{0,1,2,\dots\},\ m\ge p,\ p\in\mathbb{N}$.

The result is sharp for the function $f(z)$ given by

$$f(z)=z^{p}+\frac{2(B-A-1)p^{2}}{k(1-B)\left((\lambda-\delta)(\beta-\alpha)k+p\right)^{n}}z^{m},\quad (m\ge p). \tag{2.2}$$

Proof: Assume that the condition (2.1) is true. We must show that $f(z)\in$ $A(\alpha,\beta,\lambda,\delta,n,p,A,B)$, or equivalently prove that

$$\left|\frac{pz^{p}+z\left(D_p^n(\alpha,\beta,\lambda,\delta)f(z)\right)'}{(B-2A)pz^{p}+Bz\left(D_p^n(\alpha,\beta,\lambda,\delta)f(z)\right)'}\right|<1.$$

We have

$$\left|\frac{pz^{p}+z\left(D_p^n(\alpha,\beta,\lambda,\delta)f(z)\right)'}{(B-2A)pz^{p}+Bz\left(D_p^n(\alpha,\beta,\lambda,\delta)f(z)\right)'}\right|=$$

$$\left|\frac{pz^{p}+z\left(pz^{p-1}+\sum_{k=m}^{\infty}\frac{\left((\lambda-\delta)(\beta-\alpha)k+p\right)^{n}}{p}k\,a_{p+k}z^{k-1}\right)}{(B-2A)pz^{p}+Bz\left(pz^{p-1}+\sum_{k=m}^{\infty}\frac{\left((\lambda-\delta)(\beta-\alpha)k+p\right)^{n}}{p}k\,a_{p+k}z^{k-1}\right)}\right|$$

$$=\left|\frac{2p^{2}z^{p}+\sum_{k=m}^{\infty}\left((\lambda-\delta)(\beta-\alpha)k+p\right)^{n}k\,a_{k}z^{k}}{2(B-A)p^{2}z^{p}+B\sum_{k=m}^{\infty}\left((\lambda-\delta)(\beta-\alpha)k+p\right)^{n}k\,a_{k}z^{k}}\right|$$

$$\leq\left\{\frac{2p^{2}+\sum_{k=m}^{\infty}\left((\lambda-\delta)(\beta-\alpha)k+p\right)^{n}k\,a_{k}}{2(B-A)p^{2}+B\sum_{k=m}^{\infty}\left((\lambda-\delta)(\beta-\alpha)k+p\right)^{n}k\,a_{k}}\right\}<1.$$

The last inequality by (2.1) is true.

Conversely, suppose that $f(z)\in A(\alpha,\beta,\lambda,\delta,n,p,A,B)$. we must show that the condition (2.1) holds true, we have

$$\left|\frac{pz^p+z\left(D_p^n(\alpha,\beta,\lambda,\delta)f(z)\right)'}{(B-2A)pz^p+Bz\left(D_p^n(\alpha,\beta,\lambda,\delta)f(z)\right)'}\right|<1,$$

hence we get

$$=\left|\frac{2p^2z^p+\sum_{k=m}^{\infty}\left((\lambda-\delta)(\beta-\alpha)k+p\right)^n k\,a_k z^k}{2(B-A)p^2z^p+B\sum_{k=m}^{\infty}\left((\lambda-\delta)(\beta-\alpha)k+p\right)^n k\,a_k z^k}\right|<1.$$

Since $\operatorname{Re}(z)<|z|$, so we have

$$\operatorname{Re}\left\{\frac{2p^2z^p+\sum_{k=m}^{\infty}\left((\lambda-\delta)(\beta-\alpha)k+p\right)^n k\,a_k z^k}{2(B-A)p^2z^p+B\sum_{k=m}^{\infty}\left((\lambda-\delta)(\beta-\alpha)k+p\right)^n k\,a_k z^k}\right\}<1.$$

We choose the values of z on the real axis and letting $z\to1^{-1}$, then we obtain

$$\left\{\frac{2p^2+\sum_{k=m}^{\infty}\left((\lambda-\delta)(\beta-\alpha)k+p\right)^n k\,a_k}{2(B-A)p^2+B\sum_{k=m}^{\infty}\left((\lambda-\delta)(\beta-\alpha)k+p\right)^n k\,a_k}\right\}<1,$$

then

$$\sum_{k=m}^{\infty}k(1-B)\left((\lambda-\delta)(\beta-\alpha)k+p\right)^n a_k<2(B-A-1)p^2$$

and the proof is complete.

Corollary 2.2: Let $f(z)\in \mathrm{A}(\alpha,\beta,\lambda,\delta,n,p,A,B)$ then

$$a_k \le \frac{2(B-A-1)p^2}{k(1-B)\left((\lambda-\delta)(\beta-\alpha)k+p\right)^n} \qquad (k \ge m),$$

where $-1 \le B < A \le 1,\ \alpha \ge 0, \beta \ge 0, \lambda \ge 0, \delta \ge 0, \lambda > \delta, \beta > \alpha$.

Equality holds for the functions of the form

$$f(z) = z^p + \frac{2(B-A-1)p^2}{k(1-B)\left((\lambda-\delta)(\beta-\alpha)k+p\right)^n} z^k, \quad (k \ge m).$$

3. Distortion theorems

Theorem 3.1: Let the function $f(z)$ of the form (1.1), be in the class $A(\alpha,\beta,\lambda,\delta,n,p,A,B)$, then for $0 < |z| = r < 1$,

$$r^p - \frac{2(B-A-1)p}{(1-B)\left(p\left[(\lambda-\delta)(\beta-\alpha)+1\right]\right)^n} r^m \le |f(z)| \le r^p + \frac{2(B-A-1)p}{(1-B)\left(p\left[(\lambda-\delta)(\beta-\alpha)+1\right]\right)^n} r^m \qquad (3.1)$$

and

$$pr^{p-1} - \frac{2(B-A-1)p^2}{(1-B)\left(p\left[(\lambda-\delta)(\beta-\alpha)+1\right]\right)^n} r^m \le |f(z)|' \le pr^{p-1} + \frac{2(B-A-1)p^2}{(1-B)\left(p\left[(\lambda-\delta)(\beta-\alpha)+1\right]\right)^n} r^m \qquad (3.2)$$

The equality in (3.1) and (3.2) are attained for the function given by

$$f(z) = z^p - \frac{2(B-A-1)p^2}{(1-B)\left((\lambda-\delta)(\beta-\alpha)+p\right)^n} z^m, \quad (m \ge p) \qquad (3.3)$$

Proof: Since $m \geq p$, and $f(z) \in \mathrm{A}(\alpha,\beta,\lambda,\delta,n,p,A,B)$, in view of Theorem 2.1 , we have

$$p(1-B)\left(p\left[(\lambda-\delta)(\beta-\alpha)+1\right]\right)^{n}\sum_{k=m}^{\infty}a_k \leq \sum_{k=m}^{\infty}k(1-B)\left((\lambda-\delta)(\beta-\alpha)k+p\right)^{n}a_k < 2(B-A-1)p^2$$

Which yields

$$\sum_{k=m}^{\infty}a_k \leq \frac{2(B-A-1)p}{(1-B)\left(p\left[(\lambda-\delta)(\beta-\alpha)+1\right]\right)^{n}}$$

Consequently, for $|z|=r<1$, we obtain

$$|f(z)| \leq r^p + r^m\sum_{k=m}^{\infty}a_k \leq r^p + \frac{2(B-A-1)p}{(1-B)\left(p\left[(\lambda-\delta)(\beta-\alpha)+1\right]\right)^{n}}r^m$$

and

$$|f(z)| \geq r^p - r^m\sum_{k=m}^{\infty}a_k \geq r^p - \frac{2(B-A-1)p}{(1-B)\left(p\left[(\lambda-\delta)(\beta-\alpha)+1\right]\right)^{n}}r^m$$

which prove the assertion (3.1) of Theorem 4.1

Also from Theorem 2.1, it follows that

$$\sum_{k=m}^{\infty}k\,a_k \leq \frac{2(B-A-1)p^2}{(1-B)\left(p\left[(\lambda-\delta)(\beta-\alpha)+1\right]\right)^{n}}$$

Consequently, for $|z|=r<1$, we have

$$\begin{aligned}|f'(z)| &\le pr^{p-1}+\sum_{k=m}^{\infty} ka_k r^{k-1}\\ &\le pr^{p-1}+r^{m-1}\sum_{k=m}^{\infty} ka_k\\ &\le pr^{p-1}+\frac{2(B-A-1)p^2}{(1-B)\left(p\left[(\lambda-\delta)(\beta-\alpha)+1\right]\right)^n}r^{m-1}\end{aligned}$$

and

$$\begin{aligned}|f'(z)| &\ge pr^{p-1}-\sum_{k=m}^{\infty} ka_k r^{k-1}\\ &\ge pr^{p-1}-r^{m-1}\sum_{k=m}^{\infty} ka_k\\ &\ge pr^{p-1}-\frac{2(B-A-1)p^2}{(1-B)\left(p\left[(\lambda-\delta)(\beta-\alpha)+1\right]\right)^n}r^{m-1},\end{aligned}$$

Which prove the assertion (3.2) of Theorem 4.

Finally, it is easy to see that the bounds in (3.1) and (3.2) are attained for the function $f(z)$ given by (3.3).

Next, we deal neighborhood concepts for functions in the class $\mathrm{A}(\alpha,\beta,\lambda,\delta,n,p,A,B)$. This concept has been investigated by several authors; Goodman [6], Rusheweyh [8]. O. Altintas and S. Owa [2], see also [3],[5].

4. Neighborhoods and partial sums

Definition 4.1: Let $-1\le B<A\le 1,\ \alpha\ge 0,\beta\ge 0,\lambda\ge 0,\delta\ge 0,\lambda>\delta,\beta>\alpha$, $n\in\{0,1,2,\ldots\}$, $p\in\mathbb{N}$ and $\varepsilon>0$. We define the ε-neighborhood of a function $f(z)\in \mathrm{A}(p)$ and denote $N_{\varepsilon}(f)$ such that

$$N_{\varepsilon}(f)=\{g\in \mathrm{A}(p):\ g(z)=z^{p}+\sum_{k=m}^{\infty}b_k z^k \text{, and}$$

$$\sum_{k=m}^{\infty}\frac{k(1+B)\left((\lambda-\delta)(\beta-\alpha)k+p\right)^n}{2(B-A-1)p^2}|a_k-b_k|\le\varepsilon\Big\}.$$

Theorem 4.2: Let $\varepsilon>0$ and $f(z)\in \mathrm{A}(p)$ given by (1.1) satisfies the inclusion property:

$$\frac{f(z)+\mu z^p}{1+\mu}\in \mathrm{A}(\alpha,\beta,\lambda,\delta,n,p,A,B)\ , \tag{4.1}$$

For any complex number μ such that $|\mu|<\varepsilon$, then $N_{\varepsilon}(f)\subset \mathrm{A}(\alpha,\beta,\lambda,\delta,n,p,A,B)$.

Proof: We have $f(z)\in \mathrm{A}(\alpha,\beta,\lambda,\delta,n,p,A,B)$ for $\sigma\in\mathbb{C}$ with $|\sigma|=1$, that

$$\left[\frac{z\left(D_p^n(\alpha,\beta,\lambda,\delta)f(z)\right)'+p z^p}{(B-2A)p z^p+Bz\left(D_p^n(\alpha,\beta,\lambda,\delta)f(z)\right)'}\right]\ne\sigma$$

Equivalently, we must have

$$\frac{(f*Q)(z)}{z^p}\ne 0,\quad \left(z\in U\setminus\{0\}\right), \tag{4.2}$$

Where $Q(z)=z^p+\sum_{k=m}^{\infty}e_k z^k$ and

$$e_k\le\frac{k(1-\sigma B)\left((\lambda-\delta)(\beta-\alpha)k+p\right)^n}{2\left(\sigma(B-A)-1\right)p^2} \tag{4.3}$$

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$$|e_k| \leq \left| \frac{k(1-\sigma B)\left((\lambda-\delta)(\beta-\alpha)k+p\right)^n}{2\left(\sigma(B-A)-1\right)p^2} \right| \leq \frac{k(1+B)\left((\lambda-\delta)(\beta-\alpha)k+p\right)^n}{2(B-A-1)p^2} \tag{4.4}$$

If $f(z) \in A(p)$ given by (1.1), satisfies the inclusion property (4.1), then (4.2) yields

$$\left| \frac{(f*Q)(z)}{z^p} \right| \geq \varepsilon, \quad \left(z \in U \setminus \{0\}\right). \tag{4.5}$$

Now, if we suppose that

$$g(z) = z^p + \sum_{k=m}^{\infty} b_k z^k, \tag{4.6}$$

We easily see that

$$\left| \frac{(g-f)(z)*Q(z)}{z^p} \right| = \left| \sum_{k=m}^{\infty} (b_k - a_k) e_k z^k \right| \leq \sum_{k=m}^{\infty} |b_k - a_k| |e_k| |z|^k \leq \sum_{k=m}^{\infty} \frac{k(1+B)\left((\lambda-\delta)(\beta-\alpha)k+p\right)^n}{2(B-A-1)p^2} |b_k - a_k| \leq \varepsilon$$

then

$$\left| \frac{(g)(z)*Q(z)}{z^p} \right| = \left| \frac{f+(g-f)(z)*Q(z)}{z^p} \right| \geq \left| \frac{(f)(z)*Q(z)}{z^p} \right| - \left| \frac{(g-f)(z)*Q(z)}{z^p} \right| > 0$$

Thus, for any $\sigma \in \square$ with $|\sigma| = 1$, we have

$$\frac{(g)(z)*Q(z)}{z^p}\neq 0, \qquad (z\in U\setminus\{0\}),$$

Which implies that $g(z)\in \mathrm{A}(\alpha,\beta,\lambda,\delta,n,p,A,B)$ and finally $N_\varepsilon(f)\subset$ $\mathrm{A}(\alpha,\beta,\lambda,\delta,n,p,A,B)$.

Theorem 4.3: Let $f(z)$ be defined by (1.1) and the partial sums $S_1(z)$ and $S_q(z)$ be defined by

$S_1(z)=z^p$ and

$$S_q(z)=z^p+\sum_{k=m}^{m+q-2}a_k z^k, \qquad (q>m,\ m\geq p,\ p\in\mathbb{N}).$$

Also, suppose that $\sum_{k=m}^{\infty}C_k a_k\leq 1$, where

$$C_k=\frac{(p+k)(1-B)\left((\lambda-\delta)(\beta-\alpha)k+p\right)^n}{2(B-A-1)p^2}.$$

Then

(i) $f\in \mathrm{A}(\alpha,\beta,\lambda,\delta,n,p,A,B)$.

(ii) $$\mathrm{Re}\left\{\frac{f(z)}{S_q(z)}\right\}>1-\frac{1}{C_q}. \qquad (4.6)$$

(iii) $$\mathrm{Re}\left\{\frac{C_q(z)}{f(z)}\right\}>\frac{1}{1+C_q}, \quad z\in U,\ q>m. \qquad (4.7)$$

The estimates in (4.6) and (4.7) are sharp.

Proof: (i) Since $\dfrac{z^p+\mu z^p}{1+\mu}=z^p\in \mathrm{A}(\alpha,\beta,\lambda,\delta,n,p,A,B)$, $|\mu|<1$, then by Theorem 4.2, we have $N_1(z^p)\subset \mathrm{A}(\alpha,\beta,\lambda,\delta,n,p,A,B)$, $p\in\mathbb{N}$ ($N_1(z^p)$ denoting that 1-neighborhood).

Now since $\sum_{k=m}^{\infty}C_k a_k\le 1$ then $f\in N_1(z^p)$ and $f\in \mathrm{A}(\alpha,\beta,\lambda,\delta,n,p,A,B)$.

(ii) Since $\{C_k\}$ is an increasing sequence, we obtain

$$\sum_{k=m}^{m+q-2}a_k+C_q\le\sum_{k=q+m-1}^{\infty}a_k\le\sum_{k=m}^{\infty}C_k a_k\le 1 \tag{4.8}$$

Sitting

$$G_1(z)=C_q\left(\frac{f(z)}{S_q(z)}-\left(1-\frac{1}{C_q}\right)\right)\frac{C_q\sum_{k=q+m-1}^{\infty}a_k z^{k+p}}{1+\sum_{k=m}^{q+m-2}a_k z^{k+p}}+1,$$

from (4.8) we get

$$\left|\frac{G_1(z)-1}{G_1(z)+1}\right|=\left|\frac{C_q\sum_{k=q+m-1}^{\infty}a_k z^{k+p}}{2+2\sum_{k=m}^{q+m-2}a_k z^{k+p}+C_q\sum_{k=q+m-1}^{\infty}a_k z^{k+p}}\right|$$

$$\le\frac{C_q\sum_{k=q+m-1}^{\infty}a_k}{2-2\sum_{k=m}^{q+m-2}a_k-C_q\sum_{k=q+m-1}^{\infty}a_k}\le 1.$$

Which readily yields the assertion (4.6) of Theorem 4.3, if we take

(iii) Similarly, if we put

$$G_2(z) = (1+C_q)\left(\frac{S_q(z)}{f(z)} - \frac{C_q}{1+C_q}\right)$$

$$= 1 - \frac{(1+C_q)\sum_{k=q+m-1}^{\infty} a_k z^{k+p}}{1+\sum_{k=m}^{q+m-2} a_k z^{k+p}}$$

And make use of (4.8), we get

$$\left|\frac{G_1(z)-1}{G_1(z)+1}\right| \le \frac{(1+C_q)\sum_{k=q+m-1}^{\infty} a_k}{2-2\sum_{k=m}^{q+m-2} a_k - (1-C_q)\sum_{k=q+m-1}^{\infty} a_k} \le 1 \quad ,$$

Which yields the inequality (4.7) of the Theorem 4.3.

5. integral representation

In the next theorem , we obtain the integral representation for the $D_p^n(\alpha,\beta,\lambda,\delta)f(z)$

Theorem 5.1: Let $f(z) \in \mathrm{A}(\alpha,\beta,\lambda,\delta,n,p,A,B)$, then

$$D_p^n(\alpha,\beta,\lambda,\delta)f(z) = \int_0^z t^p \left(\frac{2(1-A\,\Phi(t))}{1-B\,\Phi(t)} - p\right) dt \ ,$$

where $|\Phi(z)| < 1$.

Proof: By putting

$$\frac{p z^p + z\left(D_p^n(\alpha,\beta,\lambda,\delta)f(z)\right)'}{2z^p} = L(z), \quad (z \in U \setminus \{0\})$$

we have

$$L(z) \prec \frac{1+Az}{1+Bz}$$

Also we can write

$$\left|\frac{L(z)-1}{BL(z)-A}\right| < 1$$

Or equivalently $\frac{L(z)-1}{BL(z)-A} = \Phi(z), \quad |\Phi(z)| < 1$. So

$$\frac{pz^p + z\left(D_p^n(\alpha,\beta,\lambda,\delta)f(z)\right)'}{2z^p} = \frac{1-A\Phi(z)}{1-B\Phi(z)}, \qquad \left(z \in U \setminus \{0\}\right)$$

which gives

$$\left(D_p^n(\alpha,\beta,\lambda,\delta)f(z)\right)' = z^p\left(\frac{2\left(1-A\Phi(z)\right)}{1-B\Phi(z)} - p\right),$$

after integration, we obtain

$$D_p^n(\alpha,\beta,\lambda,\delta)f(z) = \int_0^z t^p\left(\frac{2\left(1-A\Phi(t)\right)}{1-B\Phi(t)} - p\right)dt \ ,$$

and this gives the result.

6. Weighted mean and arithmetic mean

Definition 6.1: Let f and g be in the class $A(\alpha,\beta,\lambda,\delta,n,p,A,B)$. Then the weighted mean E_s of f and g is given by

$$E_s(z) = \frac{1}{2}\left[(1-s)f(z) + (1+s)g(z)\right], \quad 0 < s < 1.$$

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Theorem 6.2: Let f and g be in the class $\mathrm{A}(\alpha,\beta,\lambda,\delta,n,p,A,B)$. Then the weighted mean of f and g is also in the class $\mathrm{A}(\alpha,\beta,\lambda,\delta,n,p,A,B)$.

Proof: By definition 6.1, we have

$$E_s(z)=\frac{1}{2}\left[(1-s)f(z)+(1+s)g(z)\right]$$

$$E_s(z)=\frac{1}{2}\left[(1-s)\left(z^p+\sum_{k=m}^{\infty}a_k z^k\right)+(1+s)\left(z^p+\sum_{k=m}^{\infty}b_k z^k\right)\right]$$
$$=z^p+\sum_{k=m}^{\infty}\frac{1}{2}\left((1-s)a_k+(1+s)b_k\right)z^k.$$

Since f and g be in the class $\mathrm{A}(\alpha,\beta,\lambda,\delta,n,p,A,B)$, so by Theorem 2.1, we get

$$\sum_{k=m}^{\infty}k(1-B)\left((\lambda-\delta)(\beta-\alpha)k+p\right)^n a_k\le 2(B-A-1)p^2$$

and

$$\sum_{k=m}^{\infty}k(1-B)\left((\lambda-\delta)(\beta-\alpha)k+p\right)^n b_k\le 2(B-A-1)p^2.$$

Hence

$$\sum_{k=m}^{\infty}k(1-B)\left((\lambda-\delta)(\beta-\alpha)k+p\right)^n\left(\frac{1}{2}(1-s)a_k+\frac{1}{2}(1+s)b_k\right)$$
$$=\frac{1}{2}(1-s)\sum_{k=m}^{\infty}k(1-B)\left((\lambda-\delta)(\beta-\alpha)k+p\right)^n a_k$$
$$+\frac{1}{2}(1+s)\sum_{k=m}^{\infty}k(1-B)\left((\lambda-\delta)(\beta-\alpha)k+p\right)^n b_k$$
$$\le(1-s)(B-A-1)p^2+(1+s)(B-A-1)p^2$$
$$=2(B-A-1)p^2.$$

This shows $E_s \in \mathrm{A}(\alpha,\beta,\lambda,\delta,n,p,A,B)$.

Theorem 6.3: Let $f_1(z), f_2(z),\ldots, f_r(z)$ defined by

$$f_i(z)=z^p+\sum_{k=m}^{\infty}a_{k,i}z^k,\ \left(a_{k,i}\geq 0,\ i=1,2,\ldots,r,\ \ m\geq p\right), \qquad (6.1)$$

Be in the class $\mathrm{A}(\alpha,\beta,\lambda,\delta,n,p,A,B)$, then the arithmetic mean of $f_i(z)$, $(i=1,2,\ldots,r)$ defined by

$$g(z)=\frac{1}{r}\sum_{i=1}^{r}f_i(z) \qquad (6.2)$$

is also in the class $\mathrm{A}(\alpha,\beta,\lambda,\delta,n,p,A,B)$.

Proof: By using (6.1) and (6.2), we can write

$$g(z)=\frac{1}{r}\sum_{i=1}^{r}f_i(z)=\frac{1}{r}\sum_{i=1}^{r}\left(z^p+\sum_{k=m}^{\infty}a_{k,i}z^k\right)=z^p+\sum_{k=m}^{\infty}\left(\frac{1}{r}\sum_{i=1}^{r}a_{k,i}\right)z^k.$$

Since $f_i(z)\in\mathrm{A}(\alpha,\beta,\lambda,\delta,n,p,A,B)$ for every $(i=1,2,\ldots,r)$. Then by using Theorem 2.1 we prove that

$$\sum_{k=m}^{\infty}k(1-B)\big((\lambda-\delta)(\beta-\alpha)k+p\big)^n\left(\frac{1}{r}\sum_{i=1}^{r}a_{k,i}\right)$$
$$=\frac{1}{r}\sum_{i=1}^{r}\left(\sum_{k=m}^{\infty}k(1-B)\big((\lambda-\delta)(\beta-\alpha)k+p\big)^n a_{k,i}\right)$$
$$\leq\frac{2}{r}\sum_{i=1}^{r}(B-A-1)p^2=2(B-A-1)p^2.$$

The proof is complete.

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